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Measurable Sets

The **length** of an interval I, written $l(I)$, is defined to be the difference of the endpoints of the interval I. Thus, irrespective of whether an interval I with a and b as its endpoints is closed, open, open-closed or closed-open, the length $l(I)$ is $b-a$, where $a < b$. In case $a=b$, the interval $[a, b]$ degenerates to a point and has length zero while an infinite interval has length infinity. Thus, length is an example of a **set function**, i.e., a function which associates an extended real number to each set in some collection of sets. In the case of length, the domain is the collection of all intervals. The set function l clearly satisfies the following:

1. $l(I) \geqslant 0$, for all intervals I.
2. If $\{I_i\}$ is a countable collection of mutually disjoint intervals, then

$$l\left(\bigcup_{i=1}^{\infty} I_i\right) = \sum_{i=1}^{\infty} l(I_i).$$

3. If x is any fixed real number, then

$$l(I) = l(I+x).$$

In the above, we have said that in the case of length, the domain is the collection of all intervals. We would now like to extend the notion of length to more complicated and arbitrary sets than intervals.

1 LENGTH OF SETS

Let O be an open set in $\mathbf{R}$. Then O can be written as a countable union of mutually disjoint open intervals $\{I_i\}$, unique except so far as order is concerned; i.e.,

$$O = \bigcup_{i=1}^{\infty} I_i.$$

The length of the open set O **is defined by**

$$l(O)=\sum_{i=1}^{\infty} l(I_i).$$

The length $l(O)$ is well defined since the sum on the right does not depend on the order of the terms used in the summing process. Thus, the length of an open set is the sum of the lengths of the intervals (of course open and mutually disjoint) comprising O.

It is easy to verify that if O_1 and O_2 are two open sets in $\mathbb{R}$ such that $O_1 \subset O_2$, then

$$l(O_1) \leqslant l(O_2).$$

Hence, for any open set O contained in $[a, b]$, we have

$$0 \leqslant l(O) \leqslant b-a.$$

Further, let F be a closed set contained in some interval $[a, b]$. Then **the length of the closed set** F **is defined by**

$$l(F)=b-a-l(F^c),$$

where $F^c=[a, b]-F$. It can easily be seen that $l(F) \geqslant 0$.

So far, we have extended the concept of length to open and closed sets. And since the classes of these sets are too restricted, we would like to extend the concept of length to a wider class of sets in $\mathbb{R}$, if possible, to the class of all sets in $\mathbb{R}$. In this regard, we imagine a function m which assigns to each set E in $\mathbb{R}$, a nonnegative extended real number, written $m(E)$, called the *measure* of E (an extension of the notion of length function), satisfying the following properties:

1. $m(E)$ is defined for all sets $E \in \mathcal{P}(\mathbb{R})$.
2. $m(I)=l(I)$, for an interval I.
3. If $\{E_i\}$ is a sequence of disjoint sets, then

$$m\left(\bigcup_{i=1}^{\infty} E_i\right)=\sum_{i=1}^{\infty} m(E_i).$$

(This property is known as **countable additivity.**)

4. $m(E+y)=m(E)$, where y is any fixed number. (This property is known as **translation invariance.**)

Unfortunately, it is impossible to construct a set function which satisfies all the above four properties (1) to (4). In fact, if the continuum hypothesis "any uncountable set of all real numbers is equivalent to the set of all real numbers" is assumed, one cannot construct such a measure satisfying the properties (1) to (4). As a result one of these four properties must be sacrificed or weakened at least. Following Henri Lebesgue (1875-1941), who made many contributions to measure theory and integration, it is most useful to retain the last three properties, i.e., (2) to (4), and to weaken the condition given in (1) so that $m(E)$ need not be defined for all sets E in $\mathbf{R}$. Still, of course, we shall be interested in defining $m(E)$ for as many sets as possible.

Weakening property (1) is not the only approach; it is also possible to replace property (3) of countable additivity by the weaker property of finite additivity: for each finite sequence $\{E_i\}$ of disjoint sets, we have $m(\cup E_i) = \sum m(E_i)$. Another possible alternative to property (3) is countable subadditivity which is satisfied by the outer measure. Thus it is convenient to introduce first a set function, the outer measure, defined for all sets in $\mathbf{R}$ and is related to the measure of the set (when it exists).

2 OUTER MEASURE

All the sets considered in this chapter are contained in $\mathbf{R}$, unless stated otherwise. We shall be concerned particularly with intervals I of the form $]a, b[$ unless otherwise specified.

Let us consider the family $\mathcal{F}$ of all countable collections of open intervals. For any arbitrary $\mathcal{J} \in \mathcal{F}$, the sum $\sum_{I \in \mathcal{J}} l(I)$ is a non-negative extended real number. Further, this sum depends only on $\mathcal{J}$ and not on the order used in the summing process.

Now, let E be an arbitrary set. Consider the subfamily $\mathcal{C}$ of $\mathcal{F}$ consisting of countable collections $\mathcal{J}$ of open intervals $\{I_i\}$ such that $E \subset \bigcup_i I_i$; i.e.

$$\mathcal{C} = \{\mathcal{J} : \mathcal{J} \in \mathcal{F} \text{ and } \mathcal{J} \text{ covers } E\}.$$

The subfamily $\mathcal{C}$ is obviously nonempty. Thus we obtain a well defined number $m^*(E)$ in the set of all nonnegative extended real numbers given by

$$m^*(E) = \inf \{ \sum_{I \in \mathcal{J}} l(I) : \mathcal{J} \in \mathcal{C}\}.$$

2.1 Definition. *The* Lebesgue outer measure *or briefly the* outer measure $m^*(E)$ *of an arbitrary set* E *is given by*

$$m^*(E) = \inf \sum_i l(I_i),$$

where the infimum is taken over all countable collections $\{I_i\}$ of open intervals such that $E \subset \bigcup_i I_i$.

Remark. The outer measure m^* is a set function which is defined from the power set $\mathcal{P}(\mathbf{R})$ into the set of all non-negative extended real numbers.

2.2 Theorem

(a) *$m^*(A) \geqslant 0$, for all sets A.*
(b) *$m^*(\phi) = 0$.*
(c) *If A and B are two sets with $A \subset B$, then $m^*(A) \leqslant m^*(B)$.*
(This property is known as **monotonicity**.)
(d) *$m^*(A) = 0$, for every singleton set A.*
(e) *The function m^* is translation invariant, i.e., $m^*(A+x) = m^*(A)$, for every set A and for every $x \in \mathbf{R}$.*

Proof. The proofs of (a) and (b) are obvious.

(c) Let $\{I_n\}$ be a countable collection of disjoint open intervals such that $B \subset \bigcup_n I_n$. Then $A \subset \bigcup_n I_n$ and therefore

$$m^*(A) \leqslant \sum_{n=1}^{\infty} l(I_n).$$

This inequality is true for any coverings $\{I_n\}$ of B. Hence the result follows.

(d) Let $A = \{x\}$ be an arbitrary singleton set. Since

$$I_n = \left]x - \frac{1}{n},\ x + \frac{1}{n}\right[$$

is an open covering of A for each $n \in \mathbf{N}$, and $l(I_n) = \frac{2}{n}$, the result follows in view of (a).

(e) Given any interval I with end points a and b, the set $I + x$ defined by

$$I + x = \{y + x : y \in I\}$$

is clearly an interval with endpoints $a + x$ and $b + x$. Moreover,

$$l(I + x) = l(I).$$

Now, let $\epsilon > 0$ be given. Then there is a countable collection $\{I_n\}$ of open intervals such that $A \subset \bigcup_n (I_n)$ and satisfies

$$\sum_{n=1}^{\infty} l(I_n) < m^*(A) + \epsilon.$$

Clearly $A+x \subset \bigcup (I_n+x)$. Therefore

$$m^*(A+x) \leqslant \sum_{n=1}^{\infty} l(I_n+x) = \sum_{n=1}^{\infty} l(I_n) < m^*(A)+\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $m^*(A+x) \leqslant m^*(A)$. The reverse inequality follows by considering $A=(A+x)-x$ and using the above.

To answer affirmatively the question whether m^* is a generalization of the length function defined for the intervals, we prove the following theorem.

2.3 Theorem. *The outer measure of an interval is its length.*

Proof. *Case 1:* Suppose I is a closed finite interval, (say) $[a, b]$. Since, for each $\epsilon > 0$, the open interval $\left]a-\frac{\epsilon}{2},\ b+\frac{\epsilon}{2}\right[$ contains $[a, b]$, we have

$$m^*(I) \leqslant l\left(\left]a-\frac{\epsilon}{2},\ b+\frac{\epsilon}{2}\right[\right) = b-a+\epsilon.$$

This being true for each $\epsilon > 0$, we must have

$$m^*(I) \leqslant b-a = l(I).$$

To complete the proof of the result, we need to show that

$$m^*(I) \geqslant b-a. \tag{1}$$

Let $\epsilon > 0$ be given. Then there exists a countable collection $\{I_n\}$ of open intervals covering $[a, b]$ such that

$$m^*(I) > \sum_n l(I_n) - \epsilon. \tag{2}$$

By the Heine-Borel Theorem (cf. I–1.1), any collection of open intervals covering $[a, b]$ contains a finite subcollection which also covers $[a, b]$, and since the sum of the lengths of the finite subcollection is not greater than the sum of the lengths of the original collection, it suffices to establish the inequality (2) for finite collections $\{I_n\}$ which cover $[a, b]$.

Since $a \in [a, b]$ implies that $a \in \bigcup_n I_n$, there must be one of the intervals I_n which contains a. Let it be $]a_1, b_1[$. Then $a_1 < a < b_1$. If $b_1 \leqslant b$, then $b_1 \in [a, b]$, and since $b_1 \notin]a_1, b_1[$, there must be an interval $]a_2, b_2[$ in the finite collection $\{I_n\}$ such that $b_1 \in]a_2, b_2[$; that is $a_2 < b_1 < b_2$. Continuing in this manner, we get intervals $]a_1, b_1[$, $]a_2, b_2[$, . . . from the collection $\{I_n\}$ such that

$$a_i < b_{i-1} < b_i, \qquad i=1, 2, \ldots$$

where $b_0 = a$. Since $\{I_n\}$ is a finite collection, this process must terminate with some interval $]a_k, b_k[$ in the collection which is possible only when $b \in]a_k, b_k[$. Thus

$$\begin{aligned}\sum_n l(I_n) &\geqslant \sum_{i=1}^{k} l(]a_i, b_i[) \\ &= (b_k - a_k) + (b_{k-1} - a_{k-1}) + \ldots + (b_1 - a_1) \\ &= b_k - (a_k - b_{k-1}) - \ldots - (a_2 - b_1) - a_1 \\ &> b_k - a_1 \\ &> b - a,\end{aligned}$$

since $a_i - b_{i-1} < 0$, $b_k > b$ and $a_1 < a$. This, in view of (2), verifies that

$$m^*(I) > b - a - \epsilon.$$

Hence $m^*(I) \geqslant b - a$.

Case 2: Suppose I is any finite interval. Then given an $\epsilon > 0$, there exists a closed finite interval $J \subset I$ such that

$$l(J) > l(I) - \epsilon.$$

Therefore,

$$l(I) - \epsilon < l(J) = m^*(J) \leqslant m^*(I) \leqslant m^*(\bar{I}) = l(\bar{I}) = l(I)$$

$$\Rightarrow \qquad l(I) - \epsilon < m^*(I) \leqslant l(I).$$

This is true for each $\epsilon > 0$. Hence $m^*(I) = l(I)$.

Case 3: Suppose I is an infinite interval. Then given any real number $K > 0$, there exists a closed finite interval $J \subset I$ such that $l(J) = K$. Thus $m^*(I) \geqslant m^*(J) = l(J) = K$, that is $m^*(I) \geqslant K$ for any arbitrary real number $K > 0$. Hence $m^*(I) = \infty = l(I)$

3 LEBESGUE MEASURE

The outer measure, although defined for all sets in $\mathbf{R}$, does not satisfy, in general, the countable additivity . In order to have the property of countable additivity satisfied, we have to restrict the domain of definition for the function m^* to some suitable subset,

$\mathscr{M}$, of the power set $\mathscr{P}(\mathbf{R})$. The members of $\mathscr{M}$ are called **measurable sets** which are defined as follows.

3.1 Definition.* A set E is said to be **Lebesgue measurable** or briefly **measurable** if for each set A, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c). \tag{3}$$

Remark. The definition of measurability says that the measurable sets are those (bounded or unbounded) which split every set (measurable or not) into two pieces that are additive with respect to the outer measure.

Since $A = (A \cap E) \cup (A \cap E^c)$ and m^* is subadditive, we always have

$$m^*(A) \leqslant m^*(A \cap E) + m^*(A \cap E^c).$$

Thus, in order to establish that E is measurable, we need only to show, for any set A, that

$$m^*(A) \geqslant m^*(A \cap E) + m^*(A \cap E^c). \tag{4}$$

Note. The inequality (4) is often used in practice to show that a given set E is measurable and the set A in reference is called **test set** since it is used to test the measurability.

Remark. H. Lebesgue, in his investigation, did not actually use the definition given above to define measurable sets. Instead, he considered set E in the bounded interval $[a, b]$ and first defined the **interior** (or **inner**) **measure** of the set E as

$$m_*(E) = b - a - m^*(E^c).$$

He, then, called the set E to be measurable if

$$m_*(E) = m^*(E).$$

In other words, E is measurable if

$$m^*(E) = b - a - m^*(E^c). \tag{5}$$

If we let $A = [a, b]$, the equality (5) becomes

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c),$$

which is the same as (3). Thus the actual definition which Lebesgue used is a special case of (3). Since Lebesgue started with sets contained in $[a, b]$, i.e. bounded sets, appropriate modifications had to be made for unbounded sets. Such modifications, however, are not needed if Definition 3.1 is used.

3.2 Definition. The set function $m : \mathcal{M} \to \mathbf{R}^*$, obtained by restricting the set functions m^* to the subset $\mathcal{M}$ of the domain of definition $\mathcal{P}(\mathbf{R})$ of m^*; that is, $m = m^*_{|\mathcal{M}}$, is called **Lebesgue measure function** for the sets in $\mathcal{M}$.

For each $E \in \mathcal{M}$, $m(E) = m^*(E)$. The extended real number $m(E)$ is called the **Lebesgue measure** or simply **measure** of the set E.

4 PROPERTIES OF MEASURABLE SETS

4.1 Theorem

(a) *If E is a measurable set, then so is E^c.*
(b) *The sets ϕ and $\mathbf{R}$ are measurable sets.*

Proof. The proof is evident from Definition 3.1.∎

4.2 Theorem *If E has the outer measure zero, then E is a measurable set. Furthermore, every subset of E is measurable.*

Proof. Let A be any set. Then

$$m^*(A) \geqslant m^*(A \cap E) + m^*(A \cap E^c),$$

since

$$A \cap E \subset E \Rightarrow m^*(A \cap E) \leqslant m^*(E) = 0$$

and

$$A \cap E^c \subset A \Rightarrow m^*(A \cap E^c) \leqslant m^*(A).$$

Hence E is measurable. The other part follows from Problem 1.∎

5 BOREL SETS AND THEIR MEASURABILITY

Given any collection $\mathcal{C}$ of subsets of a set S, consider the family $\mathcal{F}$ of all σ-algebras each of which contains $\mathcal{C}$, and let

$$\mathcal{A} = \cap\{\Sigma : \Sigma \in \mathcal{F}\}.$$

Then one can verify that $\mathcal{A}$ is the smallest σ-algebra (unique) that contains $\mathcal{C}$, in the sense that $\mathcal{A}$ is a σ-algebra containing $\mathcal{C}$ and if Σ is any other σ-algebra containing $\mathcal{C}$, then $\mathcal{A} \subset \Sigma$. Such a collection $\mathcal{A}$ is called the σ-algebra generated by $\mathcal{C}$.

Since the intersection of a countable collection of open sets in **R** need not be open, the collection of all open sets in **R** is not a σ-algebra. This motivates us to introduce the following notion.

5.1 Definition. The σ-algebra generated by the family of all open sets in **R**, denoted by $\mathcal{B}$, is called the class of Borel sets in **R**. The sets in $\mathcal{B}$ are called Borel sets in **R**.

5.2 Examples. Each of the open sets, closed sets, $\mathcal{G}_\delta$-sets, $\mathcal{F}_\sigma$-sets, $\mathcal{G}_{\delta\sigma}$-sets, $\mathcal{F}_{\sigma\delta}$-sets, . . . is a simple type of Borel set.

The class of Borel sets plays an important role in analysis in general, and in measure theory in particular. We now prove that the Borel sets are measurable (in sense of Lebesgue).

5.3 Theorem. *Every Borel set in* **R** *is measurable; that is,* $\mathcal{B} \subset \mathcal{M}$.

Proof. We prove the theorem in several steps by using the fact that $\mathcal{M}$ is a σ-algebra.

Step 1: *The interval* $]a, \infty[$ *is measurable.*

It is enough to show, for any set A, that

$$m^*(A) \geqslant m^*(A_1) + m^*(A_2),$$

where $A_1 = A \cap]a, \infty[$ and $A_2 = A \cap]-\infty, a]$.

If $m^*(A)=\infty$, our assertion is trivially true. Let $m^*(A) < \infty$. Then, for each $\epsilon > 0$, $\exists$ a countable collection $\{I_n\}$ of open intervals that covers A and satisfies

$$\sum_{n=1}^{\infty} l(I_n) < m^*(A)+\epsilon.$$

Write $I_n' = I_n \cap]a, \infty[$ and $I_n'' = I_n \cap]-\infty, a]$. Then,

$$\begin{aligned} I_n' \cup I_n'' &= \{I_n \cap]a, \infty[\} \cup \{I_n \cap]-\infty, a]\} \\ &= I_n \cap]-\infty, \infty[\\ &= I_n, \end{aligned}$$

and $I_n' \cap I_n'' = \phi$. Therefore,

$$\begin{aligned} l(I_n) &= l(I_n') + l(I_n'') \\ &= m^*(I_n') + m^*(I_n''). \end{aligned}$$

But

$$A_1 \subset [\cup I_n] \cap]a, \infty[= \cup (I_n \cap]a, \infty[) = \cup I_n',$$

so that $m^*(A_1) \leqslant m^*(\cup I_n') \leqslant \sum m^*(I_n')$. Similarly $A_2 \subset \cup I_n''$ and so $m^*(A_2) \leqslant \sum m^*(I_n'')$. Hence,

$$\begin{aligned} m^*(A_1)+m^*(A_2) &\leqslant \sum \{m^*(I_n') + m^*(I_n'')\} \\ &= \sum l(I_n) < m^*(A)+\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this verifies the result.

Step 2: *The interval* $]-\infty, a]$ *is measurable* since

$$]-\infty, a] =]a, \infty[^c$$

Step 3: *The interval* $]-\infty, b[$ *is measurable* since it can be expressed as a countable union of the intervals of the form as in Step 2; that is,

$$]-\infty, b[= \bigcup_{n=1}^{\infty}]-\infty, b-\frac{1}{n}].$$

Step 4: Since any open interval $]a, b[$ can be expressed as

$$]a, b[=]-\infty, b[\cap]a, \infty[,$$

it is measurable.

Step 5: *Every open set is measurable*. It is so because it can be expressed as a countable union of open intervals (disjoint).

Hence, in view of Step 5, the σ-algebra $\mathscr{M}$ contains all the open sets in $\mathbb{R}$. Since $\mathscr{B}$ is the smallest σ-algebra containing all the open sets, we conclude that $\mathscr{B} \subset \mathscr{M}$. This completes the proof of the theorem. ∎

NONMEASURABLE SETS

We now turn to the question whether or not there exist sets which are nonmeasurable in the sense of Lebesgue.

Most of the sets we usually come across in analysis are measurable. However, there are several examples of nonmeasurable sets given by G. Vitali (1905), Van Vleck (1905), F. Bernstein (1908) and others. However, all these examples have been constructed on the assumption that the axiom of choice of set theory is valid, and it was not clear until recently whether a nonmeasurable set could be constructed without assuming the validity of the axiom of choice. Recently, Robert Solovay (1970) has solved this problem by proving that the existence of nonmeasurable sets cannot be established if the axiom of choice is disallowed.

In this section, we discuss an example of a nonmeasurable set which is a slight modification of the one given by Vitali. Before we do so we need certain preliminaries.

Definition. If x and y are real numbers in $[0, 1[$, then the **sum modulo 1**, $\dot{+}$, of x and y is defined by

$$x \dot{+} y = \begin{cases} x+y, & x+y < 1 \\ x+y-1, & x+y \geqslant 1. \end{cases}$$

Definition. If E is a subset of $[0, 1[$, then the **translate modulo 1** of E by y is defined to be the set given by

$$E \dot{+} y = \{z : z = x \dot{+} y,\ x \in E\}.$$

It is easy to verify that:

(i) $x, y \in [0, 1[\Rightarrow x \dot{+} y \in [0, 1[$.

(ii) The operation $\dot{+}$ is commutative and associative.

(iii) If we assign to each $x \in [0, 1[$, the angle $2\pi x$, then the sum modulo 1 corresponds to the addition of angles, and translate modulo 1 by y corresponds to rotation through an angle of $2\pi y$.

Theorem. *Let $E \subset [0,1[$ be a measurable set* and $y \in [0,1[$ *be given. Then the set $E \dot{+} y$ is measurable and $m(E \dot{+} y) = m(E)$.*

Proof. Define

$$\begin{cases} E_1 = E \cap [0, 1-y[\\ E_2 = E \cap [1-y, 1[. \end{cases}$$

Clearly E_1 and E_2 are two disjoint measurable sets such that $E_1 \cup E_2 = E$. Therefore

$$m(E) = m(E_1) + m(E_2).$$

Now, $E_1 \dot{+} y = E_1 + y$ and $E_2 \dot{+} y = E_2 + y - 1$ and so $E_1 \dot{+} y$ and $E_2 \dot{+} y$ both are measurable sets with

$$\begin{cases} m(E_1 \dot{+} y) = m(E_1 + y) = m(E_1) \\ m(E_2 \dot{+} y) = m(E_2 + y - 1) = m(E_2), \end{cases}$$

since m is translation invariant

$$\begin{aligned} E \dot{+} y &= (E_1 \cup E_2) \dot{+} y \\ &= (E_1 \dot{+} y) \cup (E_2 \dot{+} y) \end{aligned}$$

Hence $E \dot{+} y$ is a measurable set with

$$\begin{aligned} m(E \dot{+} y) &= m(E_1 \dot{+} y) + m(E_2 \dot{+} y) \\ &= m(E_1) + m(E_2) \\ &= m(E). \blacksquare \end{aligned}$$

Measurable Functions

DEFINITION

In what follows, we shall make use of the following notations:

$$E(f \geqslant \alpha) = \{x \in E : f(x) \geqslant \alpha\}$$

$$E(f = \alpha) = \{x \in E : f(x) = \alpha\}$$

$$E(f \leqslant \alpha) = \{x \in E : f(x) \leqslant \alpha\}$$

$$E(f > \alpha) = \{x \in E : f(x) > \alpha\}$$

$$E(f < \alpha) = \{x \in E : f(x) < \alpha\}.$$

Definition. An extended real-valued function* f defined on a measurable set E is said to be **Lebesgue-measurable** or, more briefly, **measurable** on E, if the set $E(f > \alpha)$ is measurable for all real numbers α.

Note. The measure of the set $E(f > \alpha)$ may be finite or infinite.

Justification of Definition 1.1. As α varies, the behaviour of the set $E(f > \alpha)$ describes how the values of the function f are distributed. Intitutively, it is obvious that the smoother the function f is, the smaller the variety of the sets will be. For instance, if $E = \mathbf{R}$ and f is continuous on $\mathbf{R}$, the set $E(f > \alpha)$ is always open.

Problem Show that a constant function with a measurable domain is measurable.

Solution. Let $f : E$ (measurable) $\to R^*$ be a constant function defined by $f(x) = K$, where K is a constant. We clearly note, for any real number α, that

$$E(f > \alpha) = \begin{cases} E & \text{if } \alpha < K \\ \phi & \text{if } \alpha \geqslant K. \end{cases}$$

This implies that $E(f > \alpha)$ is a measurable set since both the sets E and ϕ are measurable.